

# Random integral currents.

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## Abstract

For nice functions, invariant means over integral currents (certain generalized surfaces), can be uniquely defined. That may have applications to define Nambu-like string theory.

## 1 Currents (generalized surfaces).

Let  $K$  be a compact set in  $\mathbb{R}^n$ . Let  $\Omega^m$  be a space of  $C^\infty$  differential forms on  $K$ , with  $C^\infty$  norm. Space of  $m$  currents  $T_m$  is the space of continuous linear functionals on  $\Omega^m$ . Polyhedral chains are particular cases of currents, linear functionals on forms they define are integrals over polyhedral chains.

For  $\omega \in \Omega^m$ ,

$$M(\omega) = \sup_{x \in K} \|\omega(x)\|,$$

$$\text{where for } \xi \in \Lambda^m, \|\xi\| = \sup_{|\gamma| \leq 1} (\xi \cdot \gamma), \quad \gamma \text{ a simple m-vector.}$$

Let  $M(T)$  be the dual of  $M(\omega)$ ,

$$M(T) = \sup_{\omega \in \Omega^m, M(\omega) \leq 1} (T(\omega))$$

Space of normal currents  $N$  is a linear space of currents with  $M(T) + M(\partial T) < \infty$ .

Rectifiable currents  $R$  are currents which may be approximated in  $M$  semi-norm by integer Lipschitz chains, images under Lipschitz maps of polyhedral chains with integer coefficients.

Integral currents  $I$  are normal currents such that  $T$  and  $\partial T$  are rectifiable currents. Integral currents form an abelian group. We equip integral currents with flat semi-norm  $\|\cdot\|_F$

$$\|T\|_F = \inf (M(R) + M(S)), \quad T = R + \partial S, \quad R, S \text{ are rectifiable}$$

It is clear that  $\|T\|_F \leq M(T)$ .

## 2 Addition-invariant measure on integral currents $I_m$

Let  $f(X), X \in I_m$  be a bounded uniformly continuous function on space of integral  $m$ -currents  $I_m$  with flat semi-norm. Let

$$O_f = \{f(X + Y) | Y \in I_m\} \quad (1)$$

be its  $I_m$  orbit.

**Lemma 1.** *A sequence in  $O_f$  has a subsequence convergent point-wise in  $I_m$  to a bounded continuous function on  $I_m$ .*

Let  $B_\Lambda^m = \{T \in I_m | M(T) + M(\partial T) \leq \Lambda\}$ .  $B_\Lambda^m$  is compact in the flat semi-norm  $|||_F$  [1]. Using diagonal argument, it follows that a sequence in  $O_f$  has a point-wise convergent subsequence on  $I_m$ . Indeed, let  $n_1(k)$  be a subsequence uniformly convergent on  $B_\Lambda^m$ ,  $n_2(k)$  a subsequence of  $n_1$  uniformly convergent on  $B_{2\Lambda}^m, \dots, n_p(k)$  a subsequence of  $n_{p-1}(k)$  uniformly convergent on  $B_{p\Lambda}^m, \dots$ . Let  $\hat{k} \equiv n_k(k)$ . Then  $f_{\hat{k}}(X) \rightarrow h(X)$  point-wise,  $X \in I_m$ .  $\square$

**Lemma 2.** 1) *The orbit  $O_f$  is relatively compact in the weak topology.*

2) *Weakly closed convex hull of  $O_f$  is weakly compact*

A space dual to the space of continuous bounded functions on a normal topological space  $S$  is the space  $B$  of regular Borel measures on the field of closed sets, and with norm being total variation.  $I_m$  is a space with a semi-norm  $|||_F$ , and it is normal. A sequence in  $O_f$  has a point-wise convergent subsequence  $f_{\hat{k}}(X)$ , and  $f_{\hat{k}}(X)$  is uniformly bounded. By dominated convergence theorem, for any measure  $\mu \in B$ ,  $\int f_{\hat{k}}(X) d\mu$  is convergent. Therefore the orbit  $O_f$  is relatively sequentially compact in the weak topology.

2) follows from 1), see [4] V 6.4  $\square$

**Theorem 1.** *Let  $f(X), X \in I_m$  be a bounded uniformly continuous function on space of integral  $m$ -currents  $I_m$  with flat semi-norm. There is unique mean of  $f(X)$  over  $X \in I_m$ , invariant under addition in  $I_m$ . That is, there is uniquely defined constant  $\langle f \rangle$ ,  $\langle f(\cdot + Y) \rangle =$*

*$\langle f(\cdot) \rangle$ , such that for any  $\epsilon > 0$  there exists  $\left\{ \lambda_i \in \mathbb{R}, Y_i \in I_m \mid \lambda_i \geq 0, \sum_{i=1}^N \lambda_i = 1 \right\}$ , such that*

$$\sup_X \left| \sum_{i=1}^N \lambda_i f(X + Y_i) - \langle f \rangle \right| < \epsilon \quad (2)$$

$I_m$  is an abelian group and acts on continuous bounded functions by shifts; such action is distal. From Markov-Kakutani theorem [4], [5], there is a unique fixed point of the action of  $I_m$  on weakly compact convex hull of the orbit  $O_f$ .  $\square$

An easy modification of the above argument can be used to compute mean over currents with prescribed boundary, by averaging over currents with zero boundary:

**Theorem 2.** *Let  $I_m^0$  be space of integral  $m$ -currents  $T$  with zero boundary,  $\partial T = 0$ . Let  $f(X), X \in I_m^0$  be a bounded uniformly continuous function. There is unique mean of  $f(X)$  over currents in  $I_m^0$ , invariant under addition in  $I_m^0$ .*

Motivated by applications, we give an example of a family of functions for which an invariant mean can be defined:

**Proposition 1.**

Let  $k$  be a  $C^\infty$  2-form on  $\mathbb{R}^n$  with compact support, and with  $\max \{\|k\|, \|dk\|\} < \infty$ . Let

$$g_k(X) = \exp \left( i \int k \llbracket X \right) \exp (i \|X\|_F), X \in I_2. \quad (3)$$

(where  $\int k \llbracket X$  is an integral of a 2-form  $k$  over integral current  $X \in I_2$ ). Let  $G_k$  be the  $I_2$  orbit of  $g_k(X)$ ,

$$G_k = \{g_k(X + Y) | X, Y \in I_2\}. \quad (4)$$

Functions in  $G_k$  are uniformly bounded, and equicontinuous, therefore the  $I_2$  mean  $\langle g_k \rangle$  can be uniquely defined.

Indeed,  $|g_k| \leq 1$ , and

$$\begin{aligned} & |g_k(X + Y) - g_k(\tilde{X} + Y)| = \\ & \left| \exp (i \int k \llbracket (X + Y)) \exp (i \|X + Y\|_F) \left( 1 - \exp (i \int k \llbracket (X - \tilde{X})) \exp (i \|\tilde{X} + Y\|_F - i \|X + Y\|_F) \right) \right| \\ & \leq \left| \int k \llbracket (X - \tilde{X}) \right| + \|X - \tilde{X}\|_F \\ & \leq (1 + \max \{\|k\|, \|dk\|\}) \|X - \tilde{X}\|_F \end{aligned}$$

(we used that  $|1 - e^{i\alpha}| \leq |\alpha|, \alpha \in \mathbb{R}$ ).

## References

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